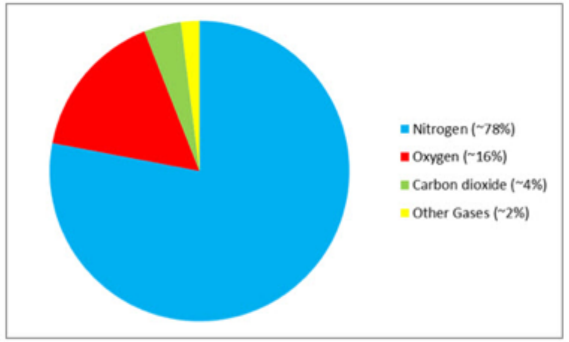

Intégration par parties (IPP)

$$\int_a^b u(x)v'(x) = \underbrace{\Big[u(x)v(x)\Big]_a^b}_{u(b)v(b)\,-\,u(a)v(a).} - \int_a^b u'(x)v(x)\,dx$$

$\sin(ax)\cdot\cos(bx)$ $a \neq b$

$e^{ax}\cdot\cos(bx)$ ou $e^{ax}\sin(bx)$

Exemple 9.17 (ii) Calculons $\int_0^{\pi} \sin(2x)\cos(3x)\,dx$

$$I = \int_0^{\pi} \sin(2x)\cos(3x)\,dx =$$

$$I = \int_0^{\pi} \sin(2x)\cos(3x)\,dx = \left[\cos(3x)\left(-\frac{1}{2}\cos(2x)\right)\right]_0^{\pi}$$

$u(x) = \cos(3x) \qquad v(x) = -\frac{1}{2}\cos(2x)$

$u' = -3\sin(3x) \qquad v'(x) = \sin(2x)$

$$\left|\ -\int_0^{\pi} \frac{3}{2}\sin(3x)\cos(2x)\,dx\right.$$

$$= -\frac{1}{2}\underbrace{\cos(3\pi)}_{-1}\underbrace{\cos(2\pi)}_{1} + \frac{1}{2}\underbrace{\cos(0)}_{1}\underbrace{\cos(0)}_{1} - \frac{3}{2}\int_0^{\pi}\sin(3x)\cos(2x)\,dx$$

$$= 1 - \frac{3}{2}\int_0^{\pi}\sin(2x)\cos(2x)\,dx = 1 - \frac{3}{2}\left[\sin(3x)\frac{1}{2}\sin(2x)\right]_0^{\pi}$$

$u(x) = \sin(3x) \qquad v(x) = \frac{1}{2}\sin(2x)$

$u' = 3\cos(3x) \qquad v'(x) = \cos(2x)$

$$\left|\ + \frac{3}{2}\int_0^{\pi} 3\cos(3x)\frac{1}{2}\sin(2x)\,dx\right.$$

$$I = 1 - \frac{3}{4} \cdot 0 \cdot 0 + \frac{3}{4} \cdot 0 \cdot 0 + \frac{9}{4} \underbrace{\int_0^{\pi} \sin(2x)\cos(3x)\,dx}_{I}$$

$$\Rightarrow I = 1 + \frac{9}{4} I \Rightarrow I\left(1 - \frac{9}{4}\right) = 1$$

$$I\left(-\frac{5}{4}\right) = 1 \Rightarrow \boxed{I = -\frac{4}{5}}$$

$\rightarrow$ ce résultat est correct (et pas celui dans le polycopié)

(iii) Calculons $\displaystyle\int_1^x \log(t)\,dt \overset{IPP}{=} \left[t \log(t)\right]_1^x - \int_1^x \frac{1}{t} \cdot t\,dt$

$u(t) = \log(t) \qquad v = t$

$u' = \frac{1}{t} \qquad v'(t) = 1$

$$= x\log(x) - \int_1^x 1\,dt = x\log(x) - x + 1$$

Théorème 9.18

Soient $a<b$, $\alpha<\beta$, $f \in C^0([a,b])$, $\varphi \in C^1([\alpha,\beta])$ tq $\varphi([\alpha,\beta]) \subseteq [a,b]$.

(i) On a $$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\varphi(x)) \cdot \varphi'(x)dx$$

"à l'aide du changement de variable $x = \varphi(y)$, calculer"

(ii) si de plus, $\varphi : [\alpha,\beta] \to [a,b]$ est bijective et $\varphi^{-1} \in C^1([a,b])$

On a $$\int_{\alpha}^{\beta} f(\varphi(x))dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} \frac{f(x)}{\varphi'(\varphi^{-1}(x))} dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)(\varphi^{-1})'(x)dx$$

Il y a une expression dans l'intégrande qui nous embête

<u>Exemples 9.19</u> : (i) Calculons $\int_0^1 \frac{1}{(1+x^2)^{3/2}} dx$ $= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx$

à l'aide du changement de variable $x = \varphi(t) = \tan(t)$

$\tan(\beta) = 1 \iff \beta = \pi/4$

$\tan(\alpha) = 0 \iff \alpha = 0$

$$\int_0^1 \frac{1}{(1+x^2)^{3/2}} dx = \int_{\tan(0)}^{\tan(\pi/4)} \frac{1}{(1+x^2)^{3/2}} dx = \int_0^{\pi/4} \frac{1}{(1+\tan^2(t))^{3/2}} \tan'(t)dt$$

$$= \int_0^{\pi/4} \frac{1}{(1+\tan^2(t))^{3/2}} \frac{1}{\cos^2(t)} dt$$

$$1+\tan^2(t) = 1 + \frac{\sin^2(t)}{\cos^2(t)} = \frac{\cos^2(t)+\sin^2(t)}{\cos^2(t)} = \frac{1}{\cos^2(t)}$$

$$= \int_0^{\pi/4} \left(\cos^2(t)\right)^{3/2} \cdot \frac{1}{\cos^2(t)}\, dt = \int_0^{\pi/4} \cos(t)\, dt$$

$$= \left[\sin(t)\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}$$

$y = \sqrt{1+x}$

(ii) Calculons $\displaystyle\int_0^3 e^{\sqrt{1+x}}\, dx = \int_1^2 e^y \cdot 2y\, dy$

$x = y^2 - 1$

$\frac{dx}{dy} = 2y$

$dx = 2y\, dy$

$$\boxed{\begin{array}{l} x = 0 \iff y = 1 \\ x = 3 \iff y = 2 \end{array}}$$

IPP

$$= \left[2ye^y\right]_1^2 - \int_1^2 2e^y\, dy = 4e^2 - 2e - 2\left[e^y\right]_1^2$$

$u(y) = 2y \qquad v(y) = e^y$

$$= 4e^2 - 2e - 2e^2 + 2e$$

$u'(y) = 2 \qquad v'(y) = e^y$

$$= 2e^2$$

Exemple 9.20 (i) Calculons $\int_1^3 \frac{1}{x(x^2+1)}\,dx$

$$\int \frac{1}{x(x^2+1)} = ?$$

$$\int^x \frac{1}{t}\,dt = \log|x| \qquad \int^x \frac{1}{1+t^2}\,dt = \arctan(t) \qquad \int^x \frac{t}{1+t^2}\,dt = \frac{1}{2}\log(1+x^2)$$

$$\frac{1}{x(x^2+1)} = A\cdot\frac{1}{x} + B\cdot\frac{x}{1+x^2} + C\cdot\frac{1}{1+x^2}$$

$$= \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2) + Bx^2 + Cx}{x(1+x^2)}$$

$$= \frac{(A+B)x^2 + Cx + A}{x(x^2+1)}$$

✓

$$1 = A + Cx + (A+B)x^2$$

$\Rightarrow \quad A = 1$ (coeff de degré 0)

$C = 0$ (coeff de degré 1)

$A+B = 0$ (coeff de degré 2)

$\Rightarrow \quad A = 1, \quad B = -1, \quad C = 0$

$$\Rightarrow \int_1^3 \frac{1}{x(x^2+1)}\,dx = \int_1^3 1\cdot\frac{1}{x} + (-1)\cdot\frac{x}{x^2+1} + 0\cdot\frac{1}{x^2+1}\,dx$$

$$= \int_1^3 \frac{1}{x}\,dx - \int_1^3 \frac{x}{x^2+1}\,dx = \Big[\log|x|\Big]_1^3 - \Big[\frac{1}{2}\log|1+x^2|\Big]_1^3$$

$$= \log(3) - \cancel{\log(1)}^{\,0} - \frac{1}{2}\log(10) + \frac{1}{2}\log(2) = \log 3 + \frac{1}{2}\log\left(\frac{1}{5}\right)$$

(ii) Calculus $\int_a^b \frac{x}{x^2+2x+3}\,dx$

$$\int^x \frac{t}{1+t^2}\,dt = \frac{1}{2}\log(1+x^2) \qquad \int^x \frac{1}{1+t^2}\,dt = \arctan(t)$$

$$x^2+2x+3 = (x+1)^2 - 1 + 3 = (x+1)^2 + 2$$

$$= 2\left(\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1\right)$$

$$\frac{dx}{dy} = \sqrt{2}$$

$$\frac{x+1}{\sqrt{2}} = y \iff x = \sqrt{2}\,y - 1$$

$$dx = \sqrt{2}\,dy$$

$$\int_a^b \frac{x}{x^2+2x+3}\,dx = \frac{1}{2}\int_a^b \frac{x}{\left(\frac{x+1}{\sqrt{2}}\right)^2 + 1}\,dx$$

$$x = a \iff y = \frac{a+1}{\sqrt{2}}$$

$$x = b \iff y = \frac{b+1}{\sqrt{2}}$$

$$= \frac{1}{2}\int_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}} \frac{\sqrt{2}\,y - 1}{y^2+1}\sqrt{2}\,dy$$

$$= \int_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}} \frac{y}{y^2+1}\, dy - \frac{\sqrt{2}}{2} \int_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}} \frac{1}{y^2+1}\, dy$$

$$= \left[\frac{1}{2} \log (1+y^2) \right]_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}} - \frac{\sqrt{2}}{2} \left[\arctan(y) \right]_{\frac{a+1}{\sqrt{2}}}^{\frac{b+1}{\sqrt{2}}}$$

(iii) Calculons $\displaystyle\int_3^5 \frac{1}{x^3-4x^2+9x-10}\, dx$

Step 1 : factoriser $x^3-4x^2+9x-10$

On cherche une racine parmi $-10, -5, -2, -1, 1, 2, 5, 10$

$x^3-4x^2+9x-10 = (x-2)(x^2-2x+5)$ $\qquad 4 - 4\cdot1\cdot5 = -16<0$

↑ div. euclidienne

Step 2 : Décomposition

$$\frac{1}{x^3-4x^2+9x-10} = \frac{A}{x-2} + \frac{Bx+C}{x^2-2x+5}$$

$$= \frac{(A+B)x^2 + (C-2A-2B)x + 5A-2C}{x^3-4x^2+9x-10}$$

$$\begin{cases} A+B=0 \\ C-2A-2B=0 \\ 5A-2C=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{5} \\ B=-\frac{1}{5} \\ C=0 \end{cases}$$

$$\frac{1}{x^3-4x^2+9x-10} = \frac{1}{5}\frac{1}{x-2} - \frac{1}{5}\frac{x}{x^2-2x+5}$$

$$\frac{x}{4\left(\frac{x-1}{2}\right)^2+4} = \frac{x}{\left(2\frac{x-1}{2}\right)^2+4}$$

$$\frac{x}{x^2-2x+5} = \frac{x}{(x-1)^2-1+5} = \frac{x}{(x-1)^2+4} = \frac{1}{4}\frac{x}{\left(\frac{x-1}{2}\right)^2+1}$$

$$\int_3^5 \frac{1}{x^3-4x^2+9x-10}dx = \frac{1}{5}\Big[\log|x-2|\Big]_3^5 - \frac{1}{20}\int_3^5 \frac{x}{\left(\frac{x-1}{2}\right)^2+1}dx$$

$$y = \frac{x-1}{2} \qquad x = 2y+1 \qquad dx = 2dy$$

$$x=3 \longleftrightarrow y=1, \qquad x=5 \longleftrightarrow y=2$$

$$= \frac{1}{5}\log(3) - \frac{1}{10}\int_1^2 \frac{2y+1}{y^2+1}dy = \frac{1}{5}\log(3) - \frac{1}{5}\int_1^2 \frac{y}{y^2+1}dy - \frac{1}{10}\int_1^2 \frac{1}{y^2+1}dy$$

$$= \frac{1}{5}\log(3) - \frac{1}{10}\left[\log(y^2+1)\right]_1^2 - \frac{1}{10}\left[\arctan(y)\right]_1^2$$

$$= \frac{1}{5}\log(3) - \frac{1}{10}\log(5) + \frac{1}{10}\log(2) - \frac{1}{10}\arctan(2) + \frac{1}{10}\arctan(1)$$

(iii) Calculons $\displaystyle\int_3^5 \frac{1}{x^3-4x^2+5x-2}\,dx$

<u>Étape 1</u> : factorisation : $x^3-4x^2+5x-2 = (x-1)^2(x-2)$

⚠ FAUX : $$\frac{1}{x^3-4x^2+5x-2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

avec cette correction c'est juste !

$$\frac{1}{x^3-4x^2+5x-2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

$$= \frac{(A+C)x^2 + (B-3A-2C)x + 2A-2B+C}{x^3-4x^2+5x-2}$$

$$\begin{cases} A+C=0 \\ B-3A-2C=0 \\ 2A-2B+C=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=-1 \\ C=1 \end{cases}$$

$$\Rightarrow \int_3^5 \frac{1}{x^3-4x^2+5x-2}\,dx = \int_3^5 (-1)\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}\,dx$$

$$= \left[-\log(x-1) + \frac{1}{x-1} + \log(x-2)\right]_3^5 = -\frac{1}{4} - \log(2) + \log(3)$$

Remarque 9.21 (i)

Cette méthode se généralise à l'intégration d'expressions

$$\frac{p(x)}{q(x)} \quad \text{avec } \deg(p) \leq \deg(q) - 1$$

Si $\deg(p) \geq \deg(q)$??

→ division euclidienne $p(x) = d(x) \cdot q(x) + r(x)$
avec $\deg(r) < \deg(q)$

$$\frac{p(x)}{q(x)} = \underbrace{d(x)} + \underbrace{\frac{r(x)}{q(x)}}$$

↳ décomposition en éléments simples.